MATH210 Spring 2015
Final Exam

1. Let $a = 4003$, $b = -127$ and $n = 85$. Compute the following:
   (a) $a + b \bmod n$
   (b) $a \cdot b \bmod n$
   (c) $a \cdot b^{-1} \bmod n$
   (d) $a^{33}b^{-33} \bmod n$

2. How many pairs $(A, B) \in \mathcal{P}(U) \times \mathcal{P}(U)$ of subsets of $U = \{1, 2, 3, 4\}$ satisfy the property $A \cup B^c = U$. For example $A = \{1, 2\}$ and $B = \{1, 2\}$ satisfy this property.

3. Find the number of integers between 1 and 10,000 inclusive that are divisible by exactly three of 3, 5, 7, 11.

4. At a party with 10 people there are 37 pairs of people which are friends.

   (a) Show that some person must have at least 8 friends.

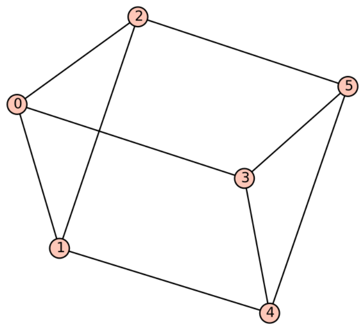
   (b) Show that some person must have less than 8 friends.

   (c) Can there be a person with no friends?

5. Establish the logical equivalence of the following two statements:
If Andy is in a good mood then either Andy is full or Andy's students understand discrete math.
If Andy is in a good mood and Andy is hungry then Andy's students understand discrete math.

6. Show the intermediate steps of **insertion sort** run on input $[4, 3, 2, 5, 1]$.

7. Use an adjacency matrix to find all walks of length 4 (order matters) in the following tent graph:



8. Prove the negation of the following: **There exists a largest prime number.**

9. The Letter People want to form a government (there are 26 letter people, 21 male consonants, and 5 female vowels, look them up later). They will elect 1 president letter and 1 vice-president letter, each of a different gender, then they elect 5 equal letters to the military council. The rankless council should always have more consonants than vowels (no vowels is a valid council). How many ways can they elect their government?

10. Suppose u_n and v_n are sequences defined recursively by

$$u_1 = 0, v_1 = 1,$$

and, for $n \geq 1$,

$$u_{n+1} = \frac{1}{2}(u_n + v_n), v_{n+1} = \frac{1}{4}(u_n + 3v_n).$$

Prove that $v_n - u_n = \frac{1}{4^{n-1}}$ for $n \geq 1$.

11. Let $A = \{a, b, c\}$.
    (a) How many equivalence relations can be defined on A?
    (b) How many partial orders can be defined on A?

12. **Dry version:** Show that $\mathbb{N} \times \mathbb{N}$ is countable.
    Cool version: The trickster convention is in town and the countable number of countably infinite hotels are all booked full of trickster guests. The guests at Andy's infinite hotel each call in a bomb threat on a distinct other hotel in town and all of those guests vacate their hotels to come stay at Andy's hotel. Show that there is room for all of them.

Extra Credit: Prove the fundamental theorem of arithmetic.